Key Points on Entropy Production as Scale-Dependent in Non-Equilibrium Field Systems

- Research suggests that entropy production ($\Pi = dS/dt$) often exhibits scale-dependent behavior in non-equilibrium systems, particularly near criticality, but this is not universally true across all domains; counterexamples exist where dissipation does not drive steeper transitions or universal scaling fails.
- It seems likely that Π modulates dynamical parameters like coupling strengths and thresholds in active matter and density field theories, though evidence leans toward context-specific rather than absolute universality.
- Evidence indicates connections between Π and criticality in some systems (e.g., active field theories), but controversy surrounds principles like maximum entropy production (MEP), with falsifications in multi-trophic ecosystems and controlled circuits.
- Cross-domain invariance of Π scaling appears limited; while observed in quantum fields and condensed matter, it does not consistently hold in biology or neural systems without additional constraints.

Overview of Frameworks

Mathematical treatments primarily come from field theories for active matter and density functionals, where Π is decomposed spectrally or via path integrals. These show Π varying with wavevector (k) or coarse-graining length (l), linking to renormalization group (RG) flows near critical points.

Scale Transformations and RG

In active systems, Π often follows power-law scaling under RG, e.g., Π(k) ~ k^{-z} for spectral modes, with exponents tied to universality classes. However, not all non-equilibrium steady states (NESS) exhibit this; some violate MEP.

Field-Theoretic Formulations

Dissipative structures and gradient flows formalize Π as functionals in Lagrangian forms, e.g., via Onsager-Machlup actions, distinguishing formal derivations from analogies.

Connections to Criticality

Π correlates with bifurcations in some models (e.g., discontinuities at fold bifurcations), but counterexamples show no link to transition steepness in wave dynamics or multi-level food webs.

Hierarchical Aspects

Π can act as a hierarchical parameter in RG treatments, renormalizing couplings, but this is model-specific.

Evidence For/Against Universality

Supporting: Spectral decompositions in active matter suggest universal scaling near criticality. Opposing: Falsifications of MEP in ecosystems and circuits indicate non-universal dissipation-criticality links.

For details, see the comprehensive survey below.

Comprehensive Survey of Mathematical Frameworks for Entropy Production in Non-Equilibrium Field Systems

This survey compiles existing mathematical structures from theoretical physics, statistical mechanics, and related fields, focusing on entropy production (Π = dS/dt) as a scale-dependent dynamical quantity. It emphasizes neutral exploration, distinguishing formal derivations from descriptive analogies, and includes evidence for/against universality. Frameworks are drawn from active matter field theories, density functional approaches, and non-equilibrium thermodynamics, with explicit equations where available. Scaling exponents, RG treatments, and connections to criticality/bifurcations are highlighted. Counterexamples and falsification arguments are addressed, revealing domains where dissipation fails to correlate with critical steepness or universal scaling.

1. Differential Equations Involving Entropy Production Modulating Dynamical Parameters

In non-equilibrium field theories, Π often enters equations modulating thresholds, couplings, or damping. For instance, in active matter models without time-reversal symmetry, Π influences phase transition steepness via spectral contributions.

- **Model: Scalar Stochastic Field Theories for Active Matter (Nardini et al., 2017)**
  Formal derivation: Π is decomposed locally (real space) or spectrally (Fourier space) in stochastic field equations for density ρ(x,t) of self-propelled particles. The Langevin equation is:

$$\partial_t \rho = \nabla \cdot (\sqrt{2D\rho}\,\xi) - \nabla \cdot (\rho \nabla \frac{\delta F}{\delta \rho}),$$

  where F is the free energy functional, D is noise strength, and ξ is Gaussian noise. Π modulates damping via irreversibility, with total rate:

$$\Pi = \sum_k \sigma_k, \quad \sigma_k = \frac{k^2}{2} \langle \phi_k \phi_{-k} \rangle,$$

  for Fourier modes φ_k. Here, Π affects critical exponents by altering correlation lengths near phase separation, e.g., modulating steepness via k-dependent terms. Distinction: Formal via path-integral derivation, not analogy.

- **Model: Non-Equilibrium Thermodynamics for Bifurcations (Yan et al., 2023)**
  Formal: In small-noise limit, Π (EPR) modulates thresholds in fold/Hopf bifurcations via:

$$\Pi = \int d\mathbf{x}\, (\mathbf{J} \cdot (D\mathbf{D})^{-1} \cdot \mathbf{J})/P,$$

  where J is probability current and P steady-state probability. Π shows discontinuities at first-order transitions (fold), modifying critical thresholds; derivatives jump at second-order (Hopf), affecting steepness. Analogy vs. derivation: Derived from Fokker-Planck, with Hamilton-Jacobi for potential φ_0.

- **Model: Fluctuating Hydrodynamics for Interacting Particles (Brossollet et al., 2025)**
  Derivation: Dean's SPDE for density ρ:

$$\partial_t \rho = -\nabla \cdot \mathbf{J} + \eta, \quad \mathbf{J} = -\rho \nabla \frac{\delta F}{\delta \rho} + f,$$

  with entropy production:

$$\dot{S} = -\frac{1}{T} \int dx\, f(x)\rho(x) \left[(\rho * \nabla V)(x) + T \nabla \rho(x) \rho(x) - f(x) \right] P_{ss}[\rho].$$

  Π modulates couplings via non-conservative forces f, affecting phase steepness in NESS. Expressed via correlations G_1, G_2.

2. Scale Transformation Laws of Entropy Production

Scaling often emerges via coarse-graining or RG, but universality is debated.

- **Scaling in Active Disordered Media (Cocconi et al., 2022)**
  Formal: Under coarse-graining length l, Π(l) scales as:

$$\Pi(l) \sim l^{d-\theta},$$

  where θ is a model-dependent exponent (e.g., θ = 2 in passive systems, modified in active). For active Model B:

$$\Pi(l) \propto l^{d-4} \int dq\, q^3 C(q),$$

  with C(q) ~ q^{-z} near criticality. Exponent α = d - θ for Π(λL) = λ^α Π(L).

- **RG Treatments with Dissipation**: In active field theories (e.g., Toner-Tu model), RG flows renormalize dissipation terms, altering Π scaling. For KPZ equation:

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \sqrt{D}\xi,$$

  active λ term leads to δD ~ ∫ dk / k^2 (UV divergent), implying RG-scale dependence of Π. In state-space RG for reaction networks (Yu et al., 2022), Π determines fixed points.

- **Scaling in NESS**: In multistable systems (Endres, 2017), Π selects states but scales with system size N as Π ~ N log N in some limits, not universal.

3. Field-Theoretic Formulations

- **Dissipative Structures (Prigogine et al.)**: Analogy-based, with instability via:

$$\delta^2 S = -\int (\delta X_i \delta J_i) dV < 0,$$

  for fluctuations δX_i, δJ_i. Entropy functional in Lagrangian: S = ∫ L dt, but often descriptive.

- **Gradient Flow Systems**: Formal in non-reciprocal Cahn-Hilliard (NRCH):

$$\partial_t \phi = -\Gamma \frac{\delta F}{\delta \phi} + W,$$

  with Π = β ∫ ∂_t φ W, breaking symmetry.

- **Onsager-Machlup Path-Integral**: In density fields:

$$A[\rho] = \frac{1}{4T} \int dt dx \left(\frac{1}{\rho} \nabla^{-1} \dot{\rho} - \nabla^{-1} \dot{\rho} - \rho(\nabla \delta F / \delta \rho - f) \right)^2 .$$

  Formal derivation for Π.

4. Cross-Domain Invariance

- Supporting: In quantum fields (e.g., vacuum fluctuations), Π scales ~ k^{-2} (similar to condensed matter active models). In biology (neural systems), Π ~ synaptic strengths, but inconsistent in multi-trophic webs.
- Opposing: No invariance in chemistry (e.g., controlled circuits) or biology (food webs >1 level).

5. Formal Connections

- To Criticality: Π peaks near Hopf, discontinuous at fold (Yan et al.).
- To Bifurcations: Via EPR in Wilson-Cowan: dS/dt modulates logistic-like transitions.
- To Landau: Non-eq. free energy F = E - D S, with expansions analogous to Landau for order parameters.
- Distinction: Formal in field theories; analogy in MEP.

6. Entropy Production as Hierarchical Parameter

In RG, Π renormalizes couplings (e.g., δD in KPZ), modifying thresholds hierarchically.

7. Falsification Arguments and Counterexamples

- Against MEP Universality: In ecosystems (del Jesus et al., 2010), MEP fails for >1 trophic level; state with max Π unstable.
- Counterexample: Resistor circuit (Dewar, 2010) under feedback; Π not maximized.
- Domains without Dissipation-Critical Steepness Correlation: Wave breaking (e.g., spectral parameter WSP); dissipation independent of steepness in some hydrodynamic regimes.
- Falsification: Ginzburg-Landau counterexample (Brazhnyi, 2014) shows stable states not at max Π.

Tables of Models and Scaling

Table 1: Key Models with Equations

Model	Equation for Π	Modulation	Derivation/Analogy
Active Scalar Fields	Π = ∑k *(k^2 / 2) <φ_k φ*{-k}>	Damping, exponents	Formal (path-integral)
Density Field DDFT	\dot{S} = - (1/T) ∫ f ρ [...] P_ss	Coupling, thresholds	Formal
Bifurcation Thermodynamics	Π = ∫ (J · (D D)^{-1} · J)/P	Thresholds, steepness	Formal
MEP (Earth Systems)	σ = F · (1/T_b - 1/T_a)	None (analogy fails)	Analogy

Table 2: Scaling Exponents and RG

System	Scaling Law/Exponent	RG Treatment
Active Disordered	Π(l) ~ l^{d-θ}, θ~2-4	Coarse-graining RG
KPZ Active	δD ~ ∫ dk/k^2, α~ -2	Perturbative RG
NESS Multistable	Π ~ N log N (system size)	None

Evidence supports scale-dependence in active/quantum domains but not universally; falsifications highlight context-specificity.

Key Citations:

- Nardini et al. (2017). Phys. Rev. X.
- Brossollet et al. (2025). arXiv:2507.15131.
- Yu et al. (2022). Phys. Rev. E.
- Yan et al. (2023). Commun. Phys.
- Cocconi et al. (2022). Phys. Rev. E.
- del Jesus et al. (2010). Philos. Trans. R. Soc. B.
- Dewar (2010). Entropy.